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- Assuming Newtonian gravity $\quad G = 6.674e{-11}\ \frac{Nm^2}{kg^2}$

position $\quad r_i(t)$

velocity $\quad v_i(t) = \dot{r}_i(t) \qquad i = 1,2,3 \qquad r_{ij} = |r_i - r_j|$

1.

a)

i) newtonian potential energy

$$\sum U_g = \sum -\frac{G m_1 m_2}{r_{12}}$$

$$U_{12} + U_{13} + U_{23} + U_{21} + U_{31} + U_{32}$$

since $\vec{r}_{ij} = |\vec{r}_i - \vec{r}_j| \equiv |\vec{r}_j - \vec{r}_i|$ then $U_{12} = U_{21}$

$$\sum U_g = 2(U_{12} + U_{13} + U_{23})$$

$$\sum U_g = 2\left(\frac{-G m_1 m_2}{\vec{r}_{12}} - \frac{G m_1 m_3}{\vec{r}_{13}} - \frac{G m_2 m_3}{\vec{r}_{23}}\right)$$

$$\sum U_g = 2\sum_{i}^{3}\sum_{j>i}^{3} \frac{-G m_i m_j}{\vec{r}_{ij}}$$

ii) Calc force & write eq of motion

$$F = -\nabla U \quad \& \quad \vec{r} = \sqrt{x^2+y^2+z^2}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle -2\left(\frac{G m_1 m_2}{\vec{r}_{12}} + \frac{G m_1 m_3}{\vec{r}_{13}} + \frac{G m_2 m_3}{\vec{r}_{23}}\right)$$

$$\nabla \frac{1}{r} = \left\langle \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$(x^2+1+z)^{-1/2}$$

$$-\tfrac{1}{2}\cdot 2x(x^2+1+z)^{-3/2} \qquad \nabla \vec{r} = \frac{-\vec{r}}{|\vec{r}|^3}$$

$$\left\langle \frac{-x}{r^3}, \frac{-y}{r^3}, \frac{-z}{r^3} \right\rangle = \frac{-1}{r^3}\hat{r}$$

$$\vec{F}_{net} = -\nabla U = +2\left(\frac{G m_1 m_2}{|\vec{r}_{12}|^3}\vec{r}_{12} + \frac{G m_1 m_3}{|\vec{r}_{13}|^3}\vec{r}_{13} + \frac{G m_2 m_3}{|\vec{r}_{23}|^3}\vec{r}_{23}\right)$$

$F = m \cdot \ddot{\vec{r}}$

mass 1 $\quad 2\left(\frac{Gm_1m_2}{|\vec{r}_{12}|^3}\vec{r}_{12} + \frac{Gm_1m_3}{|\vec{r}_{13}|^3}\vec{r}_{13}\right) = m_1\ddot{\vec{r}}_1$

mass 2 $\quad 2\left(\frac{Gm_2m_1}{|\vec{r}_{21}|^3}\vec{r}_{21} + \frac{Gm_2m_3}{|\vec{r}_{23}|^3}\vec{r}_{23}\right) = m_2\ddot{\vec{r}}_2$

mass 3 $\quad 2\left(\frac{Gm_3m_2}{|\vec{r}_{32}|^3}\vec{r}_{32} + \frac{Gm_3m_1}{|\vec{r}_{31}|^3}\vec{r}_{31}\right) = m_3\ddot{\vec{r}}_3$

iii) total energy and energy conservation

$E = T + U \qquad T_i = \frac{1}{2}m_iv_i^2$

$E = \sum_i \frac{1}{2}m_i\dot{\vec{r}}_i^{\,2} - 2G\sum_{i<j}^{3}\frac{m_im_j}{\vec{r}_{ij}}$ $\qquad$ if $\frac{dE}{dt} = 0$ energy is conserved

$$\frac{dE}{dt} = \sum_i \frac{1}{2}m_i\frac{d}{dt}(\dot{\vec{r}}_i)^2 + \sum_{i<j}\frac{d}{dt}U(\vec{r})$$

$$= \sum_i \frac{1}{2}m_i\frac{d}{dt}(\dot{\vec{r}}_i)^2 + \sum_{i<j}\frac{dU}{d\vec{r}}\frac{d\vec{r}}{dt}$$

$$= \sum_i \frac{1}{2}m_i\frac{d}{dt}(\dot{\vec{r}})^2 + \sum_{i<j}\frac{dU}{d\vec{r}}\dot{\vec{r}}$$

$$= \sum_i m_i\dot{\vec{r}}\ddot{\vec{r}} + \sum_{i<j}\frac{dU}{d\vec{r}}\dot{\vec{r}}$$ $\qquad F = -\nabla U = -\frac{d}{d\vec{r}}U$

$F = ma = m\ddot{\vec{r}}$

$m\ddot{\vec{r}} = -\frac{dU}{d\vec{r}}$

$$= \sum_i \dot{\vec{r}}\left(-\frac{dU}{d\vec{r}}\right) + \sum_{i<j}\frac{dU}{d\vec{r}}\dot{\vec{r}}$$

$\frac{dE}{dt} = 0 \qquad$ Conserved!

B) i) $P = \sum_i m_iv_i = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + m_3\dot{\vec{r}}_3$

$\dot{P} = \frac{d}{dt}(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + m_3\dot{\vec{r}}_3) = m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 + m_3\ddot{\vec{r}}_3$ $\qquad$ Sub in $F_i = m_i\ddot{\vec{r}}_i$

$\dot{P} = 2(\vec{F}_{g_{12}} + \vec{F}_{g_{13}}) + 2(\vec{F}_{g_{21}} + \vec{F}_{g_{23}}) + 2(\vec{F}_{g_{31}} + \vec{F}_{g_{32}})$

by N3L $F_{ij} = -F_{ji}$

all cancel out

$\dot{P} = 0$

if $\dot{P} = 0$ when $\dot{P} = ma$ then $a = 0$ $\ddot{\vec{r}} = 0$ $\qquad$ frames with no acceleration are inertial

$$R_{cm} = \frac{\sum_i m_i r_i}{\sum m_i} \quad \text{so} \quad R_{\substack{center\\of\\mass}} = \sum_i \vec{r}_i$$

$$P = \sum m_i \dot{r}_i = \dot{R}_{cm} \sum m_i$$

$$\dot{P} = \ddot{R}_{cm} \sum m_i \qquad \ddot{R}_{cm} = 0 \text{ because } \dot{P} = 0$$

accel in center of mass frame is zero ∴ inertial

ii) $L = \sum_i (r \times m_i \vec{v}_i) \to L_{tot} = \sum_i m_i (\vec{r}_i \times \dot{\vec{r}}_i)$

$$\dot{L}_{tot} = \sum_i m_i \left(\frac{d}{dt}\vec{r}_i \times \frac{d}{dt}\dot{\vec{r}}_i\right)$$

$$= \sum_i m_i (\dot{\vec{r}}_i \times \ddot{\vec{r}}_i)$$

in Cent of Mass frame $\sum \vec{r}_i = R_{com}$

$\sum \dot{\vec{r}}_i = \dot{R}_{com}$

$\sum \ddot{\vec{r}}_i = \ddot{R}_{com}$

and $\ddot{R}_{com} = 0$

$$\dot{L}_{tot} = \dot{R}_{com} \times \ddot{R}_{com} \left(\sum_i m_i\right)$$

$$\dot{L}_{tot} = \dot{R}_{com} \times 0 \left(\sum_i m_i\right) = 0$$ angular momentum is conserved

c) Planarity of motion

i) if $\dot{L} = 0$ then L is const. prove that $\sum_i \dot{\vec{r}}_i \cdot \vec{L} = 0$ where $\vec{L} = \sum_i m_i (\vec{r}_i \times \dot{\vec{r}}_i)$

$\sum_i \vec{r}_i \cdot \vec{L} = 0$

$$\dot{\vec{r}}_i \cdot \left(\sum m_i \vec{r}_i \times \dot{\vec{r}}_i\right)$$

$$\sum m_i \left(\vec{r}_i \cdot \vec{r}_i \times \dot{\vec{r}}_i\right)$$

know that for $\vec{a} \cdot \vec{a} \times \vec{b}$, $\vec{a} \times \vec{b}$ will produce a vector orthogonal to both $\vec{a}$ and $\vec{b}$, thus dot prod. w/ vect orthogonal to $\vec{a}$ and $\vec{a}$ will yield 0. $\vec{a} \times \vec{b} = \vec{c}$ where $\vec{c}$ orthog to $\vec{a}$, thus $\vec{a} \cdot \vec{c} = 0$

$$= \sum m_i (0)$$

$$0 = \sum m_i (\vec{r}_i \cdot \vec{r}_i \times \dot{\vec{r}}_i) = \vec{r}_i \cdot \vec{L}$$

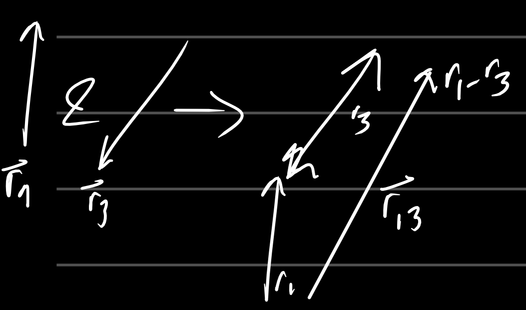
If all positions & velocities are coplanar, then all L where $L = m\vec{r}\times\dot{\vec{r}}$ would be orthogonal to that plane. Because $\dot{\vec{L}} = 0$, $\vec{L}$ is constant, thus the plane in which all positions & velocities are in stays constant.

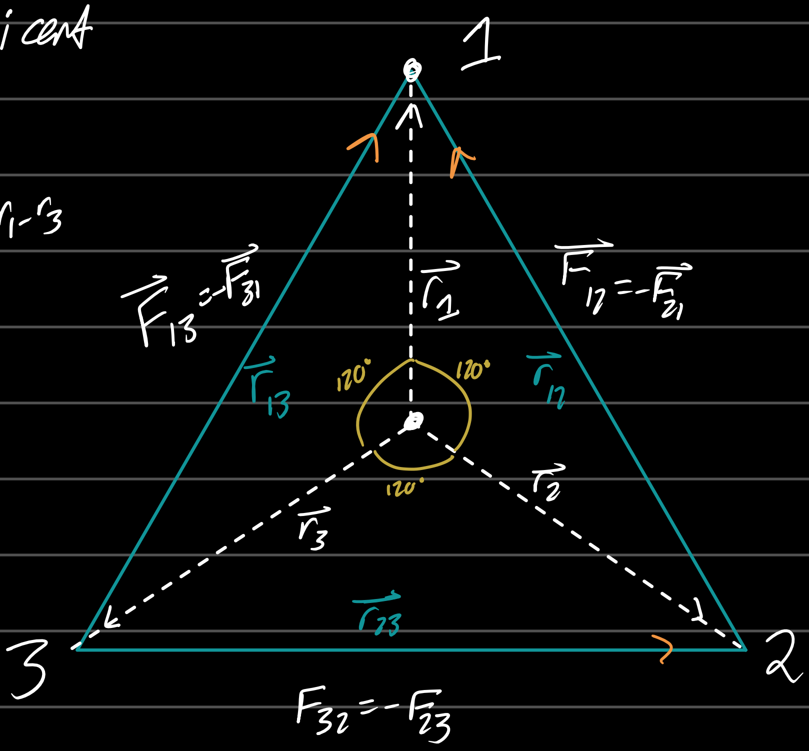
ii) $L = 0$

When $\vec{L} = \vec{0}$ then $\sum m_i \vec{r}_i \times \dot{\vec{r}}_i = 0$ thus $\vec{r}_i$ and $\dot{\vec{r}}_i$ are all parallel or anti parallel, which inherently means they all exist in one plane or that for $\vec{r}_j \times \vec{r}_i$, $i \neq j$ all the terms cancel out, which is also all coplanar.

D) Central Configurations & equilateral triangle

$\vec{a} \propto k\vec{r}_{i\,\text{cent}}$





F_{ij} : force of i onto j

Know:

$$F_{12} = \frac{-Gm_1m_2}{\vec{r}_{12}}$$

$$F_{13} = \frac{Gm_1m_3}{\vec{r}_{13}}$$

$$F_{23} = \frac{Gm_1m_2}{\vec{r}_{23}}$$

$$F_{\text{net on }1} = F_{31} + F_{21} = -\frac{Gm_1m_3}{\vec{r}_{13}} - \frac{Gm_1m_2}{\vec{r}_{12}} = -\left(\frac{Gm_1m_3}{\vec{r}_1} - \frac{Gm_1m_3}{\vec{r}_3}\right) - \left(\frac{Gm_1m_2}{\vec{r}_1} - \frac{Gm_1m_2}{\vec{r}_2}\right)$$

$$= -\frac{2Gm_1m_3}{\vec{r}_1} + \left(\frac{Gm_1m_3}{\vec{r}_3} + \frac{Gm_1m_2}{\vec{r}_2}\right)$$

$\vec{r}_3 = \langle |r_3|\cos 210, |r_3|\sin 210\rangle$

$\vec{r}_2 = \langle |r_2|\cos 330, |r_2|\sin 330\rangle$

where $|r_1| = |r_2| = |r_3|$

and $\cos 210 = -\cos 330$ & $\sin 210 = \sin 330$

so $\vec{r}_2 + \vec{r}_3 = \langle \cos 210 + (-\cos 210), 2\sin 210\rangle$

$|r_2|\langle 0, 2(\tfrac{1}{2})\rangle$

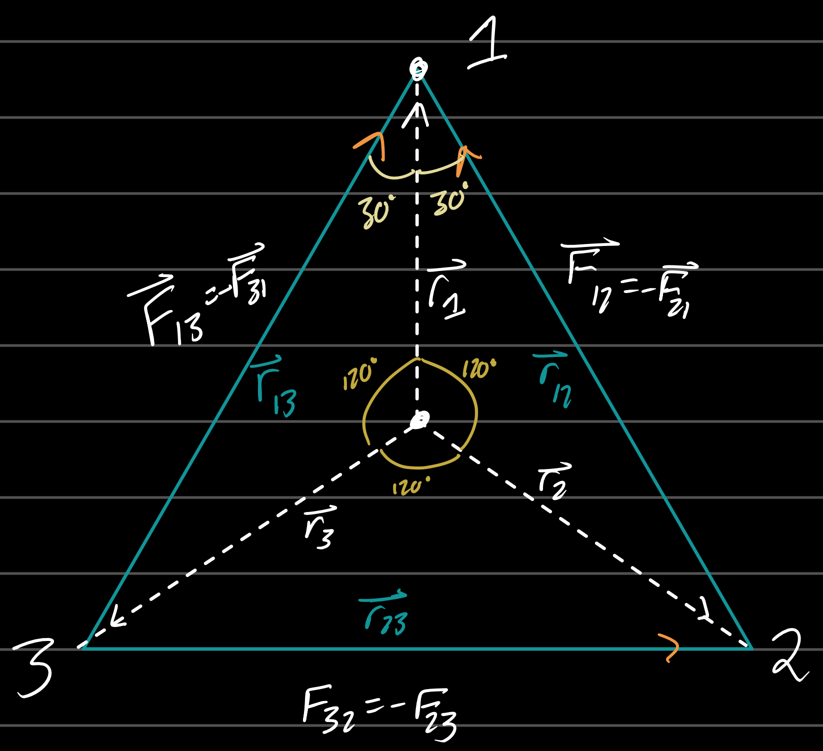
$\vec{r}_2 + \vec{r}_3 = -\hat{r}_1$

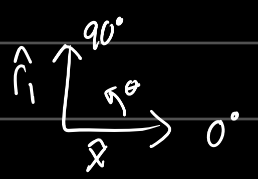
$$-\frac{2Gm_1m_3}{\vec{r}_1} + \left(\frac{Gm_1m_3}{\vec{r}_3} + \frac{Gm_1m_2}{\vec{r}_2}\right)$$

$$= -\frac{2Gm_1m_3}{\vec{r}_1} + (Gm_1m_3 + Gm_1m_2)\frac{|\vec{r}_1|}{\vec{r}_1}$$

$$-(2Gm_1m_3 + |r_1|Gm_1m_3 + |r_1|Gm_1m_2)\frac{1}{\vec{r}_1}$$

$$m_1\ddot{\vec{r}}_1 = -(2Gm_1m_3 + |r_1|Gm_1m_3 + |r_1|Gm_1m_2)\frac{1}{\vec{r}_1} \Rightarrow \ddot{\vec{r}}_1 = \lambda\vec{r}$$





1. Equilateral, so $r_1=r_2=r_3$ from com and $|r_{13}|=|r_{23}|=|r_{12}|=|r|$

2. $F_g = \frac{Gm_im_j}{|r_{ij}|^2}\vec r_{ij} \Rightarrow F_g = \frac{Gm_im_j}{|r|^2}\vec r_{ij}$

$F_g \propto \lambda \vec r_{ij}$

For mass 1, which experiences $\vec F_{21_g}$ & $\vec F_{31_g}$

$\vec F_{21_g} = -\langle |F_{21_g}|\cos 300, |F_{21_g}|\sin 300\rangle$

$\vec F_{31_g} = -\langle |F_{31_g}|\cos 240, |F_{21_g}|\sin 240\rangle$

we know $\cos 300 = -\cos 240$
$\sin 300 = \sin 240$

$\sum F_{mass\,1} = \vec F_{21_g} + \vec F_{31_g} = -\langle |F_{21_g}|\cos 300, |F_{21_g}|\sin 300\rangle - \langle |F_{31_g}|\cos 240, |F_{21_g}|\sin 240\rangle$

$\langle -|F_{21_g}|\cos 300 - |F_{31_g}|\cos 240, -|F_{21_g}|\sin 300 - |F_{31_g}|\sin 240\rangle$

$\langle -|F_{21_g}|\cos 300 + |F_{31_g}|\cos 300, -|F_{21_g}|\sin 300 - |F_{31_g}|\sin 300\rangle$

$\sum F_{mass\,1} = \langle 0, -(|F_{21_g}|+|F_{31_g}|)\sin 300\rangle$

$m_1\ddot{\vec r}_1 = (|F_{12_g}|+|F_{13_g}|)\sin 300\ \hat r_1$

Remember
$F_g = \gamma \vec r$

$m_1\ddot{\vec r}_1 = (|F_{12_g}|+|F_{13_g}|)\sin 300\ \frac{|r_1|}{\vec r_1}$

$m\ddot{\vec r}_1 = \left(\left(\frac{c_1}{r}+\frac{const_2}{r}\right)\sin 300\right)\frac{|r_1|}{\vec r_1} = \left((const_1+const_2)\sin 300\right)\vec r\,\frac{|r|}{\vec r} = m\ddot{\vec r}$

$\ddot{\vec r} = \frac{\sin 300\,(const_1+const_2)}{m}|r|$

$\ddot{\vec r} \propto |r|$ accel proportional to position!

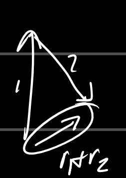
Works for both other points too

Geometrically:

$r_{13}+r_{23} = r_1-r_3+r_2-r_3$

$(r_1+r_2)-2r_3$

r_1+r_2:



anti parallel to $\vec r_3$

ii) would be homographic if $|\ddot{\vec{r}}_1| = |\ddot{\vec{r}}_2| = |\ddot{\vec{r}}_3|$

just proved $\ddot{\vec{r}}_i = \lambda \vec{r}_i$ where $|r_i| = |r_1| = |r_2| = |r_3|$ because equilateral

so in $|\ddot{\vec{r}}_i| = \lambda |\vec{r}_i|$ for any i $|\ddot{\vec{r}}_i| = \lambda |r|$

thus $|\ddot{\vec{r}}_i| = |\ddot{\vec{r}}_j|$ $j \neq i$

all accels are the same thus all points motion is the same

If $L = 0$ theres no angular rotation, thus the triangle not only keeps its shape but doesn't spin, thus maintains similar placement

Computation

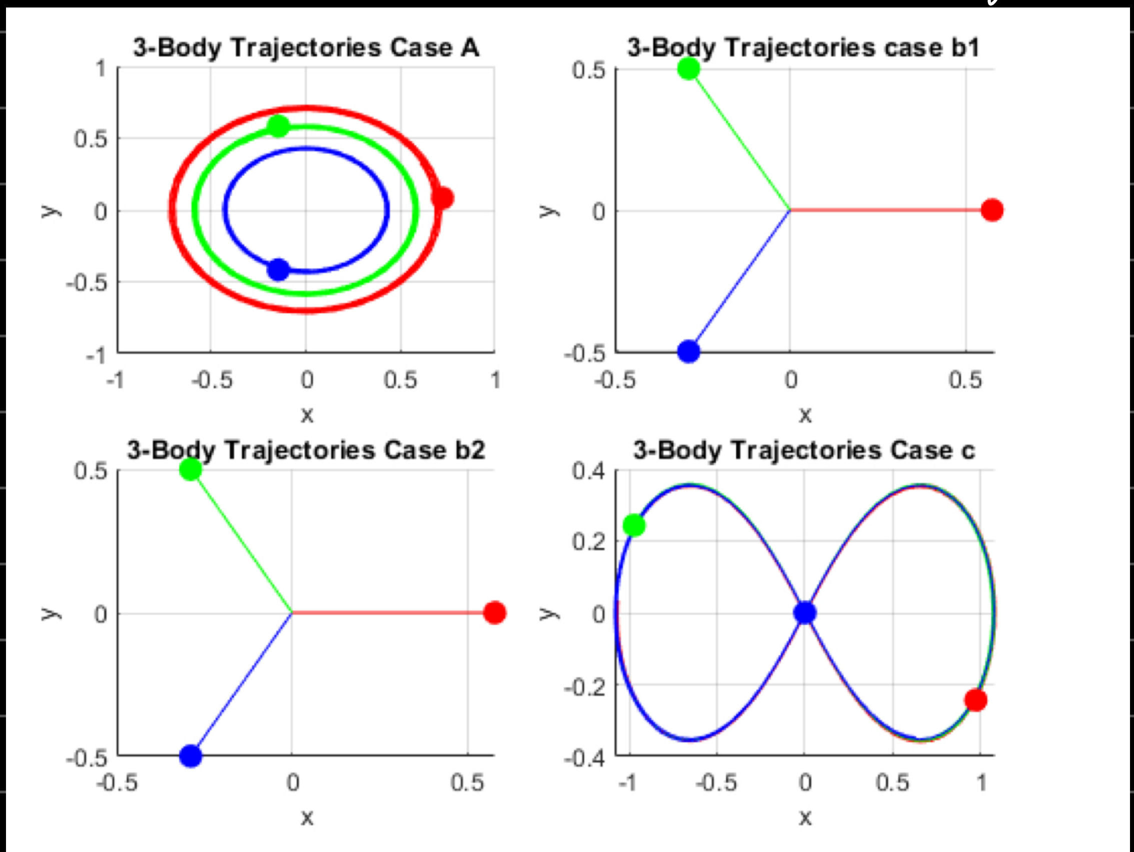
have an instantaneous pos, velocity, and accel

$$-\left(\frac{Gm_1m_2}{|\vec{r}_{12}|^3}\vec{r}_{12} + \frac{Gm_1m_3}{|\vec{r}_{13}|^3}\vec{r}_{13}\right) = m_1\ddot{\vec{r}}_1$$

$$\ddot{\vec{r}}_i m_i = -\sum_{i \neq j} \frac{Gm_im_j}{|\vec{r}_{ij}|^3}\vec{r}_{ij}$$

$$\ddot{\vec{r}}_i = -\sum_{i \neq j} \frac{Gm_j}{|r_{ij}|^3}\vec{r}_{ij}$$ turn into function

solve the diff eq to find $\vec{r}_i(t)$ for any t !



Case A: three bodies orbit the COM very slowly expanding their orbits

Case B1: the objects approach eachother fastly then slower and slower

Case B2: same as B1 but slower at start because less initial velocity, less KE_i

Case C: pretty tight consistent figure 8 orbit between 3 bodies